

Determinant representations for form factors in quantum integrable models with $GL(3)$ -invariant R -matrix

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Abstract

We obtain determinant representations for the form factors of the monodromy matrix entries in quantum integrable models solvable by the nested algebraic Bethe ansatz and possessing $GL(3)$ -invariant R -matrix. These representations can be used for the calculation of correlation functions in the models of physical interest.

Keywords: nested algebraic Bethe ansatz, scalar products, form factors.

1 Introduction

The form factor approach is one of the most effective methods for calculating correlation functions of quantum integrable models. Therefore, finding explicit and compact representations for the form factors is an important task. Currently there are several methods to study form factors of integrable systems. One of the first to be developed was the so called ‘form factor

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bootstrap approach', which has been successfully applied to integrable quantum field theory [1, 2, 3, 4, 5, 6, 7]. This method is closely related to the one based on the conformal field theory and its perturbation [8, 9, 10, 11]. It is also worth mentioning the approach developed in [12, 13, 14], where the form factors were studied via the representation theory of quantum affine algebras. All the methods listed above deal with quantum integrable models in infinite volume. Form factors in the models of finite volume were studied in [15, 16] by the algebraic Bethe ansatz [17, 18, 19, 20]. In particular, this method was found to be very efficient for quantum spin chain models, for which the solution of the quantum inverse scattering problem is known [16, 21]. Determinant representations for form factors obtained in this framework were successfully used for the calculation of correlation functions [22, 23, 24, 25].

The results listed above mostly concern the models based on $GL(2)$ symmetry or its q -deformation. Models with a higher rank symmetry were much less studied. At the same time such models play an important role in various applications. For instance, integrability has proved to be a very efficient tool for the calculation of scattering amplitudes in super-Yang-Mills theories. The calculation of these amplitudes can be related to scalar products of Bethe vectors. In particular, in the $SU(3)$ subsector of the theory, one just needs the $SU(3)$ Bethe vectors. Hence, the knowledge of the form factors is very essential in this context.

Form factors of integrable models with symmetries of high rank also appear in condensed matter physics, in particular in two-component Bose (or Fermi) gas and in the study of models of cold atoms (for e.g. ferromagnetism or phase separation). One can also mention 2-band Hubbard models (mostly in the half-filled regime), in the context of strongly correlated electronic systems. In that case, the symmetry increases when spin and orbital degrees of freedom are supposed to play a symmetrical role, leading to an $SU(4)$ or even an $SO(8)$ symmetry (see e.g. [26, 27]). All these studies require to look for integrable models with $SU(N)$ symmetry, the first step being the $SU(3)$ case. In this context it is worth mentioning the work [28], where the form factors in the model of two-component Bose gas were studied.

In this article we give determinant representations for form factors in $GL(3)$ -invariant quantum integrable models solvable by the nested algebraic Bethe ansatz [29, 30, 31]. More precisely, we calculate matrix elements of the monodromy matrix entries $T_{ij}(z)$ between on-shell Bethe vectors (that is, the eigenstates of the transfer matrix). The determinant representations given in this paper are based on the formulas obtained in [32, 33, 34]. There, however, we had slightly different representations for the form factors of the diagonal entries $T_{ii}(z)$ and the ones for $T_{ij}(z)$ with $|i - j| = 1$. Furthermore, in the case of the operators $T_{ii}(z)$ one had to consider two different cases depending on whether two Bethe vectors coincided or were different. In this paper we give more uniform determinant representations for all form factors. We also announce determinant formulas for the form factors of the operators $T_{13}(z)$ and $T_{31}(z)$. To derive these formulas, we used a new approach, which requires a detailed description. It will be given in a separate publication.

The paper is organized as follows. In section 2 we introduce the model under consideration. In section 3 we recall the results for form factors in the models with $GL(2)$ -symmetries. In section 4 we present the main results of our paper. The methods of their derivation are briefly described in section 5. In particular, we introduce there the notion of twisted transfer matrix, which appears to be very effective for the calculation of form factors of the diagonal entries. In section 6 we present a proof of some determinant representations given in section 4. Section 7 is

devoted to the discussions of some perspectives. Appendix collect several summation identities needed for the proof of the determinant representations.

2 Bethe vectors and form factors

In this section we describe the model under consideration, introduce necessary notations and define the object of our study.

2.1 Generalized $GL(3)$ -invariant model

The models considered below are described by a $GL(3)$ -invariant R -matrix acting in the tensor product of two auxiliary spaces $V_1 \otimes V_2$, where $V_k \sim \mathbb{C}^3$, $k = 1, 2$:

$$R(x, y) = \mathbf{I} + g(x, y)\mathbf{P}, \quad g(x, y) = \frac{c}{x - y}. \quad (2.1)$$

In the above definition, \mathbf{I} is the identity matrix in $V_1 \otimes V_2$, \mathbf{P} is the permutation matrix that exchanges V_1 and V_2 , and c is an arbitrary nonzero constant.

The monodromy matrix $T(w)$ satisfies the algebra

$$R_{12}(w_1, w_2)T_1(w_1)T_2(w_2) = T_2(w_2)T_1(w_1)R_{12}(w_1, w_2). \quad (2.2)$$

Equation (2.2) holds in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, where $V_k \sim \mathbb{C}^3$, $k = 1, 2$, are the auxiliary linear spaces, and \mathcal{H} is the Hilbert space of the Hamiltonian of the model under consideration. The matrices $T_k(w)$ act non-trivially in $V_k \otimes \mathcal{H}$.

The trace in the auxiliary space $V \sim \mathbb{C}^3$ of the monodromy matrix, $\text{tr} T(w)$, is called the transfer matrix. It is a generating functional of integrals of motion of the model. The eigenvectors of the transfer matrix are called on-shell Bethe vectors (or simply on-shell vectors). They can be parameterized by sets of complex parameters satisfying Bethe equations (see section 2.3).

2.2 Notations

We use the same notations and conventions as in the paper [33]. Besides the function $g(x, y)$ we also introduce a function $f(x, y)$

$$f(x, y) = 1 + g(x, y) = \frac{x - y + c}{x - y}. \quad (2.3)$$

Two other auxiliary functions will be also used

$$h(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{x - y + c}{c}, \quad t(x, y) = \frac{g(x, y)}{h(x, y)} = \frac{c^2}{(x - y)(x - y + c)}. \quad (2.4)$$

Due to the obvious property $g(-x, -y) = g(y, x)$ all the functions introduced above possess similar properties:

$$f(-x, -y) = f(y, x), \quad h(-x, -y) = h(y, x), \quad t(-x, -y) = t(y, x). \quad (2.5)$$

Before giving a description of the Bethe vectors we formulate a convention on the notations. We denote sets of variables by bar: \bar{w} , \bar{u} , \bar{v} etc. Individual elements of the sets are denoted by subscripts: w_j , u_k etc. Notations \bar{u}_i , \bar{v}_i mean $\bar{u} \setminus u_i$, $\bar{v} \setminus v_i$ etc.

In order to avoid too cumbersome formulas we use shorthand notations for products of functions g , f , and h . Namely, if these functions depend on sets of variables, this means that one should take the product over the corresponding set. For example,

$$h(z, \bar{w}) = \prod_{w_j \in \bar{w}} h(z, w_j); \quad g(u_i, \bar{u}_i) = \prod_{\substack{u_j \in \bar{u} \\ u_j \neq u_i}} g(u_i, u_j); \quad f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k). \quad (2.6)$$

We will also use a special notation $\Delta'_n(\bar{x})$ and $\Delta_n(\bar{x})$ for the products

$$\Delta'_n(\bar{x}) = \prod_{j < k}^n g(x_j, x_k), \quad \Delta_n(\bar{x}) = \prod_{j > k}^n g(x_j, x_k). \quad (2.7)$$

2.3 Bethe vectors

Now we pass to the description of Bethe vectors. A generic Bethe vector is denoted by $\mathbb{B}^{a,b}(\bar{u}; \bar{v})$. It is parameterized by two sets of complex parameters $\bar{u} = u_1, \dots, u_a$ and $\bar{v} = v_1, \dots, v_b$ with $a, b = 0, 1, \dots$. Dual Bethe vectors are denoted by $\mathbb{C}^{a,b}(\bar{u}; \bar{v})$. They also depend on two sets of complex parameters $\bar{u} = u_1, \dots, u_a$ and $\bar{v} = v_1, \dots, v_b$. The state with $\bar{u} = \bar{v} = \emptyset$ is called a pseudovacuum vector $|0\rangle$. Similarly the dual state with $\bar{u} = \bar{v} = \emptyset$ is called a dual pseudovacuum vector $\langle 0|$. These vectors are annihilated by the operators $T_{ij}(w)$, where $i > j$ for $|0\rangle$ and $i < j$ for $\langle 0|$. At the same time both vectors are eigenvectors for the diagonal entries of the monodromy matrix

$$T_{ii}(w)|0\rangle = \lambda_i(w)|0\rangle, \quad \langle 0|T_{ii}(w) = \lambda_i(w)\langle 0|, \quad (2.8)$$

where $\lambda_i(w)$ are some scalar functions. In the framework of the generalized model, $\lambda_i(w)$ remain free functional parameters. Actually, it is always possible to normalize the monodromy matrix $T(w) \rightarrow \lambda_2^{-1}(w)T(w)$ so as to deal only with the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \quad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}. \quad (2.9)$$

If the parameters \bar{u} and \bar{v} of a Bethe vector² satisfy a special system of equations (Bethe equations), then it becomes an eigenvector of the transfer matrix (on-shell Bethe vector). The system of Bethe equations can be written in the following form:

$$\begin{aligned} r_1(u_i) &= \frac{f(u_i, \bar{u}_i)}{f(\bar{u}_i, u_i)} f(\bar{v}, u_i), & i &= 1, \dots, a, \\ r_3(v_j) &= \frac{f(\bar{v}_j, v_j)}{f(v_j, \bar{v}_j)} f(v_j, \bar{u}), & j &= 1, \dots, b, \end{aligned} \quad (2.10)$$

and we recall that $\bar{u}_i = \bar{u} \setminus u_i$ and $\bar{v}_j = \bar{v} \setminus v_j$.

²For simplicity here and below we do not distinguish between vectors and dual vectors.

If \bar{u} and \bar{v} satisfy the system (2.10), then

$$\text{tr } T(w) \mathbb{B}^{a,b}(\bar{u}; \bar{v}) = \tau(w|\bar{u}, \bar{v}) \mathbb{B}^{a,b}(\bar{u}; \bar{v}), \quad \mathbb{C}^{a,b}(\bar{u}; \bar{v}) \text{tr } T(w) = \tau(w|\bar{u}, \bar{v}) \mathbb{C}^{a,b}(\bar{u}; \bar{v}), \quad (2.11)$$

where

$$\tau(w) \equiv \tau(w|\bar{u}, \bar{v}) = r_1(w)f(\bar{u}, w) + f(w, \bar{u})f(\bar{v}, w) + r_3(w)f(w, \bar{v}). \quad (2.12)$$

Remark. Observe that the system of Bethe equations (2.10) is equivalent to the statement that the function $\tau(w|\bar{u}, \bar{v})$ (2.12) has no poles in the points $w = u_i$ and $w = v_j$.

Form factors of the monodromy matrix entries are defined as

$$\mathcal{F}_{a,b}^{(i,j)}(z) \equiv \mathcal{F}_{a,b}^{(i,j)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) T_{ij}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (2.13)$$

where both $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors, and

$$\begin{aligned} a' &= a + \delta_{i1} - \delta_{j1}, \\ b' &= b + \delta_{j3} - \delta_{i3}. \end{aligned} \quad (2.14)$$

We use here superscripts B and C in order to distinguish the sets of parameters entering these two vectors. In other words, unless explicitly specified, the variables $\{\bar{u}^B; \bar{v}^B\}$ in $\mathbb{B}^{a,b}$ and $\{\bar{u}^C; \bar{v}^C\}$ in $\mathbb{C}^{a,b}$ are not supposed to be related. The parameter z is an arbitrary complex number. Acting with the operator $T_{ij}(z)$ on $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ via formulas obtained in [35] we reduce the form factor to a linear combination of scalar products, in which $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ is an on-shell vector.

2.4 Relations between form factors

Obviously, there exist nine form factors of $T_{ij}(z)$ in the models with $GL(3)$ -invariant R -matrix. However, not all of them are independent. In particular, due to the invariance of the R -matrix under transposition with respect to both spaces, the mapping³

$$\psi : T_{ij}(u) \mapsto T_{ji}(u) \quad (2.15)$$

defines an antimorphism of the algebra (2.2). Acting on the Bethe vectors this antimorphism maps them into the dual ones and vice versa

$$\psi(\mathbb{B}^{a,b}(\bar{u}; \bar{v})) = \mathbb{C}^{a,b}(\bar{u}; \bar{v}), \quad \psi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \mathbb{B}^{a,b}(\bar{u}; \bar{v}). \quad (2.16)$$

Therefore we have

$$\psi(\mathcal{F}_{a,b}^{(i,j)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)) = \mathcal{F}_{a',b'}^{(j,i)}(z|\bar{u}^B, \bar{v}^B; \bar{u}^C, \bar{v}^C), \quad (2.17)$$

where a' and b' are defined in (2.14). Hence, the form factor $\mathcal{F}_{a,b}^{(i,j)}(z)$ can be obtained from $\mathcal{F}_{a,b}^{(j,i)}(z)$ by means of the replacements $\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}$ and $\{a, b\} \leftrightarrow \{a', b'\}$.

³For simplicity we denoted mappings (2.15), (2.16) and (2.17) acting on the operators, vectors and form factors by the same letter ψ . The same is applied to the mappings (2.18), (2.19) and (2.20).

One more relationship between form factors arises due to the mapping φ :

$$\varphi : T_{ij}(u) \mapsto T_{4-j, 4-i}(-u), \quad (2.18)$$

that defines an isomorphism of the algebra (2.2) [35]. This isomorphism implies the following transform of Bethe vectors:

$$\varphi(\mathbb{B}^{a,b}(\bar{u}; \bar{v})) = \mathbb{B}^{b,a}(-\bar{v}; -\bar{u}), \quad \varphi(\mathbb{C}^{a,b}(\bar{u}; \bar{v})) = \mathbb{C}^{b,a}(-\bar{v}; -\bar{u}). \quad (2.19)$$

Since the mapping φ connects the operators T_{11} and T_{33} , it also leads to the replacement of functions $r_1 \leftrightarrow r_3$. Thus,

$$\varphi(\mathcal{F}_{a,b}^{(i,j)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B)) = \mathcal{F}_{b,a}^{(4-j, 4-i)}(-z|-\bar{v}^C, -\bar{u}^C; -\bar{v}^B, -\bar{u}^B) \Big|_{r_1 \leftrightarrow r_3}. \quad (2.20)$$

Altogether we are left with (at most) four independent form factors, for example, the form factors of the operators $T_{11}(z)$, $T_{22}(z)$, $T_{12}(z)$ and $T_{13}(z)$.

3 Form factors in $GL(2)$ -based models

Before giving the main results of this paper we recall the determinant representations for form factors obtained previously in the integrable models with $GL(2)$ -invariant R -matrix [15, 16]. Actually these results can be treated as a particular cases of form factors in the models with $GL(3)$ -invariant R -matrix, which correspond to special Bethe vectors with $a = 0$ or $b = 0$. Below we set for definiteness $b = 0$. Let

$$\mathbb{C}^a(\bar{u}) = \mathbb{C}^{a,0}(\bar{u}; \emptyset), \quad \mathbb{B}^a(\bar{u}) = \mathbb{B}^{a,0}(\bar{u}; \emptyset). \quad (3.1)$$

The Bethe vectors (3.1) become on-shell, if the parameters \bar{u} satisfy the system of Bethe equations

$$r_1(u_i) = \frac{f(u_i, \bar{u}_i)}{f(\bar{u}_i, u_i)} = (-1)^{a-1} \frac{h(u_i, \bar{u})}{h(\bar{u}, u_i)}, \quad i = 1, \dots, a. \quad (3.2)$$

Then

$$(T_{11}(w) + T_{22}(w))\mathbb{B}^a(\bar{u}) = \tau_2(w|\bar{u})\mathbb{B}^a(\bar{u}), \quad \mathbb{C}^a(\bar{u})(T_{11}(w) + T_{22}(w)) = \tau_2(w|\bar{u})\mathbb{C}^a(\bar{u}), \quad (3.3)$$

where

$$\tau_2(w) \equiv \tau_2(w|\bar{u}) = r_1(w)f(\bar{u}, w) + f(w, \bar{u}). \quad (3.4)$$

The form factors of the monodromy matrix entries in the $GL(2)$ -based models are defined as

$$\mathcal{F}_a^{(i,j)}(z) \equiv \mathcal{F}_a^{(i,j)}(z|\bar{u}^C; \bar{u}^B) = \mathbb{C}^{a'}(\bar{u}^C)T_{ij}(z)\mathbb{B}^a(\bar{u}^B), \quad (3.5)$$

where both vectors are on-shell. For conciseness, we have used the notation $a' = a + j - i$.

All the representations for the form factors of the operators $T_{ij}(z)$, $i, j = 1, 2$, are based on the determinant formula for the scalar product of on-shell Bethe vector and generic Bethe vector

[36]. This formula immediately implies such the representations for $\mathcal{F}_a^{(1,2)}(z)$ and $\mathcal{F}_a^{(2,1)}(z)$. Namely, let $\bar{x} = \{\bar{u}^B, z\}$. Then

$$\mathcal{F}_a^{(1,2)}(z) = \Delta'_{a'}(\bar{u}^C) \Delta_{a'}(\bar{x}) \det_{a'} n_{jk}, \quad (3.6)$$

where

$$n_{jk} = \frac{c}{g(x_k, \bar{u}^C)} \frac{\partial \tau_2(x_k | \bar{u}^C)}{\partial u_j^C}. \quad (3.7)$$

The result for $\mathcal{F}_a^{(2,1)}(z)$ can be obtained from (3.6), (3.7) via the replacements $\bar{u}^C \leftrightarrow \bar{u}^B$ and $a' \leftrightarrow a$:

$$\mathcal{F}_a^{(2,1)}(z) = \Delta'_a(\bar{u}^B) \Delta_a(\bar{y}) \det_a \left(\frac{c}{g(y_k, \bar{u}^B)} \frac{\partial \tau_2(y_k | \bar{u}^B)}{\partial u_j^B} \right), \quad (3.8)$$

where $\bar{y} = \{\bar{u}^C, z\}$.

There exist several equivalent formulas for form factors of the diagonal entries $T_{ss}(z)$, $s = 1, 2$. Here we give representations in the form of determinants of matrices of the size $(a+1) \times (a+1)$. We have

$$\mathcal{F}_a^{(s,s)}(z) = \Delta'_a(\bar{u}^C) \Delta_{a+1}(\bar{x}) \det_{a+1} n_{jk}^{(s)}, \quad s = 1, 2, \quad (3.9)$$

where $\bar{x} = \{\bar{u}^B, z\}$. The entries $n_{jk}^{(s)}$ of the matrices $n^{(s)}$ in the first a rows ($j = 1, \dots, a$) are given by (3.7). Pay attention, however, that the cardinality of the set \bar{u}^C in (3.7) is equal to $a+1$, while we have $\#\bar{u}^C = a$ for the form factors $\mathcal{F}_a^{(s,s)}(z)$. One can say that in both cases $\#\bar{u}^C = a'$. In the last row one has

$$\begin{aligned} n_{a+1,k}^{(1)} &= (-1)^a r_1(x_k) h(\bar{u}^B, x_k), \\ n_{a+1,k}^{(2)} &= h(x_k, \bar{u}^B). \end{aligned} \quad (3.10)$$

Remark. Observe that due to Bethe equations (3.2) we have $n_{a+1,k}^{(1)} + n_{a+1,k}^{(2)} = 0$ for $k = 1, \dots, a$ (that is, if $x_k \in \bar{u}^B$). Therefore the form factor of the transfer matrix $T_{11}(z) + T_{22}(z)$ reduces to the eigenvalue $\tau_2(z | \bar{u}^B)$ multiplied by the scalar product of the vectors $\mathbb{C}^a(\bar{u}^C)$ and $\mathbb{B}^a(\bar{u}^B)$. This result, of course, immediately follows from the definition of on-shell Bethe vectors.

Making the replacement $\bar{u}^C \leftrightarrow \bar{u}^B$ in (3.9)–(3.10) we obtain alternative determinant representations for form factors of the operators $T_{ss}(z)$. In spite of these two types of representations look very different, one can prove their equivalence (see e.g. [37]).

Thus, we see that in the $GL(2)$ -based models the form factors of the monodromy matrix entries are proportional to the Jacobians of the eigenvalue $\tau_2(w)$ on the left or right Bethe vector (up to possible modification of one row).

4 Main results

The results given in section 3 suggest their possible generalization to the models with $GL(3)$ -invariant R -matrix. Indeed, it seems quite reasonable to expect that form factors of the monodromy matrix entries in such models also are proportional to the Jacobians of the transfer

matrix eigenvalue. However, this conjecture confirms only partly. In this section we show that the form factors of the operators $T_{ij}(z)$ in the $GL(3)$ -based models have more sophisticated determinant representations.

4.1 Form factors of off-diagonal elements

The determinant representations for form factors of the operators $T_{ij}(z)$ with $|i - j| = 1$ have the most simple structure. They were calculated in [34]. We start our exposition with the form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$:

$$\mathcal{F}_{a,b}^{(1,2)}(z) \equiv \mathcal{F}_{a,b}^{(1,2)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) T_{12}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (4.1)$$

where both $\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are on-shell Bethe vectors. As in the $GL(2)$ case, we used a' and b' notation, whose definition depends on the form factor we are considering. For the $\mathcal{F}_{a,b}^{(1,2)}(z)$ form factor, we have $a' = a + 1$, $b' = b$.

In order to describe the determinant representation for this form factor we introduce a set of variables $\bar{x} = \{x_1, \dots, x_{a'+b}\}$ as the union of three sets $\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$, and define a scalar function $\mathcal{H}_{a',b}$ as

$$\mathcal{H}_{a',b} = \frac{h(\bar{x}, \bar{u}^B)h(\bar{v}^C, \bar{x})}{h(\bar{v}^C, \bar{u}^B)} \Delta'_{a'}(\bar{u}^C) \Delta'_b(\bar{v}^B) \Delta_{a+b+1}(\bar{x}). \quad (4.2)$$

Proposition 4.1. ([34]) *The form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$ admits the following determinant representation:*

$$\mathcal{F}_{a,b}^{(1,2)}(z) = \mathcal{H}_{a',b} \det_{a'+b} \mathcal{N}, \quad (4.3)$$

where $(a' + b) \times (a' + b)$ matrix \mathcal{N} has the following entries

$$\mathcal{N}_{j,k} = \frac{c}{f(x_k, \bar{u}^B)f(\bar{v}^C, x_k)} \frac{g(x_k, \bar{u}^B)}{g(x_k, \bar{u}^C)} \frac{\partial \tau(x_k|\bar{u}^C, \bar{v}^C)}{\partial u_j^C}, \quad j = 1, \dots, a', \quad (4.4)$$

$$\mathcal{N}_{a'+j,k} = \frac{-c}{f(x_k, \bar{u}^B)f(\bar{v}^C, x_k)} \frac{g(\bar{v}^C, x_k)}{g(\bar{v}^B, x_k)} \frac{\partial \tau(x_k|\bar{u}^B, \bar{v}^B)}{\partial v_j^B}, \quad j = 1, \dots, b. \quad (4.5)$$

We see that this representation involves two eigenvalues of the transfer matrix. Namely, the elements in the first $a + 1$ rows of the matrix \mathcal{N} are proportional to the derivatives of the eigenvalue $\tau(x_k|\bar{u}^C, \bar{v}^C)$ on the left vector, while the elements in the last b rows of the matrix \mathcal{N} are proportional to the derivatives of the eigenvalue $\tau(x_k|\bar{u}^B, \bar{v}^B)$ on the right vector. Thus, as we have mentioned in the beginning of the section, this determinant representation is not a straightforward generalization of the formula (3.6). Nevertheless, one can easily see that at $b = 0$ the equation (4.3) reproduces the result (3.6).

Determinant representations for other form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i - j| = 1$ can be derived from (4.3) by the mappings (2.17), (2.20). First, we give the explicit formulas for the form factor of the operator T_{23}

$$\mathcal{F}_{a,b}^{(2,3)}(z) \equiv \mathcal{F}_{a,b}^{(2,3)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) T_{23}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (4.6)$$

where now for the $\mathcal{F}_{a,b}^{(2,3)}(z)$ form factor, we have $a' = a$ and $b' = b + 1$.

We introduce a set of variables $\bar{y} = \{y_1, \dots, y_{a+b'}\}$ as the union of three sets $\bar{y} = \{\bar{u}^C, \bar{v}^B, z\}$ and a function

$$\tilde{\mathcal{H}}_{a,b'} = \frac{h(\bar{y}, \bar{u}^C)h(\bar{v}^B, \bar{y})}{h(\bar{v}^B, \bar{u}^C)} \Delta'_a(\bar{u}^B) \Delta'_{b'}(\bar{v}^C) \Delta_{a+b+1}(\bar{y}). \quad (4.7)$$

Proposition 4.2. ([34]) *The form factor $\mathcal{F}_{a,b}^{(2,3)}(z)$ admits the following determinant representation:*

$$\mathcal{F}_{a,b}^{(2,3)}(z) = \tilde{\mathcal{H}}_{a,b'} \det_{a+b'} \tilde{\mathcal{N}}, \quad (4.8)$$

where $(a + b') \times (a + b')$ matrix $\tilde{\mathcal{N}}$ has the following entries

$$\tilde{\mathcal{N}}_{j,k} = \frac{c}{f(y_k, \bar{u}^C)f(\bar{v}^B, y_k)} \frac{g(y_k, \bar{u}^C)}{g(y_k, \bar{u}^B)} \frac{\partial \tau(y_k | \bar{u}^B, \bar{v}^B)}{\partial u_j^B}, \quad j = 1, \dots, a, \quad (4.9)$$

$$\tilde{\mathcal{N}}_{a+j,k} = \frac{-c}{f(y_k, \bar{u}^C)f(\bar{v}^B, y_k)} \frac{g(\bar{v}^B, y_k)}{g(\bar{v}^C, y_k)} \frac{\partial \tau(y_k | \bar{u}^C, \bar{v}^C)}{\partial v_j^C}, \quad j = 1, \dots, b'. \quad (4.10)$$

Using (2.5) it is easy to check that the representation for $\mathcal{F}_{a,b}^{(2,3)}(z)$ can be obtained from the one for $\mathcal{F}_{a,b}^{(1,2)}(z)$ via the following replacements

$$\bar{u}^C \leftrightarrow -\bar{v}^C, \quad \bar{u}^B \leftrightarrow -\bar{v}^B, \quad r_1 \leftrightarrow r_3, \quad a \leftrightarrow b, \quad (4.11)$$

as it is prescribed by the isomorphism (2.20).

At the same time, one can observe that the formulas for these two form factors are also related by the replacements

$$\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}, \quad \{a, b\} \leftrightarrow \{a', b'\}. \quad (4.12)$$

Be careful however that in doing these transformations, the definition of a' and b' changes when going from $\mathcal{F}_{a,b}^{(1,2)}(z)$ to $\mathcal{F}_{a,b}^{(2,3)}(z)$ (and vice-versa).

Applying mapping (2.17) to representations (4.3), (4.8) we arrive at the following

Proposition 4.3. ([34]) *The form factor $\mathcal{F}_{a,b}^{(3,2)}(z)$ admits the following determinant representation:*

$$\mathcal{F}_{a,b}^{(3,2)}(z) = \mathcal{H}_{a',b} \det_{a'+b} \mathcal{N}, \quad (4.13)$$

where $\mathcal{H}_{a',b}$ and \mathcal{N} are given by (4.2) and (4.3) respectively.

The form factor $\mathcal{F}_{a,b}^{(2,1)}(z)$ admits the following determinant representation:

$$\mathcal{F}_{a,b}^{(2,1)}(z) = \tilde{\mathcal{H}}_{a,b'} \det_{a+b'} \tilde{\mathcal{N}}, \quad (4.14)$$

where $\tilde{\mathcal{H}}_{a,b'}$ and $\tilde{\mathcal{N}}$ are given by (4.7) and (4.8) respectively.

Remark. We would like to stress again that although the representations (4.13) and (4.14) formally coincide with (4.3) and (4.8), the values of a' and b' in these formulas are different. Indeed, one has $a' = a + 1$ and $b' = b$ in (4.3), while $a' = a$ and $b' = b - 1$ in (4.13). Similarly $a' = a$ and $b' = b + 1$ in (4.8), while $a' = a - 1$ and $b' = b$ in (4.14). Therefore, in particular, the matrices \mathcal{N} and $\tilde{\mathcal{N}}$ in (4.3) and (4.8) have a size $(a + b + 1) \times (a + b + 1)$, while in the equations (4.13) and (4.14) the same matrices have a size $(a + b) \times (a + b)$.

4.2 Form factors of diagonal elements

The form factors of diagonal entries of the monodromy matrix

$$\mathcal{F}_{a,b}^{(s,s)}(z) \equiv \mathcal{F}_{a,b}^{(s,s)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ss}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (4.15)$$

were calculated in [33]. Here we give different representations for them. In a sense they are analogous to the determinant formulas for form factors in the $GL(2)$ -based models (see section 3). Namely, they are based on the determinant of the matrix \mathcal{N} (4.4), (4.5), but one row of this matrix should be modified.

As before we combine the sets \bar{u}^B and \bar{v}^C and the parameter z into the set $\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$. We also introduce three $(a+b+1)$ -component vectors $Y^{(s)}$, $s = 1, 2, 3$, as

$$\begin{aligned} Y_k^{(s)} &= \delta_{s2} - \delta_{s1} + \frac{u_k^B}{c}(\delta_{s1} - \delta_{s3}) \left(\frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)} - 1 \right), \quad k = 1, \dots, a; \\ Y_{a+k}^{(s)} &= \delta_{s2} - \delta_{s3} + \frac{v_k^C + c}{c}(\delta_{s1} - \delta_{s3}) \left(\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)} - 1 \right), \quad k = 1, \dots, b. \end{aligned} \quad (4.16)$$

In these formulas δ_{sk} are Kronecker deltas. The values of $Y_{a+b+1}^{(s)}$ are crucial only in the case when $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$, that is $\bar{u}^B = \bar{u}^C = \bar{u}$ and $\bar{v}^B = \bar{v}^C = \bar{v}$. We define them as

$$Y_{a+b+1}^{(1)} = \frac{r_1(z)f(\bar{u}, z)}{f(\bar{v}, z)f(z, \bar{u})}, \quad Y_{a+b+1}^{(2)} = 1, \quad Y_{a+b+1}^{(3)} = \frac{r_3(z)f(z, \bar{v})}{f(\bar{v}, z)f(z, \bar{u})}. \quad (4.17)$$

One can set here $\bar{v} = \bar{v}^C$ or $\bar{v} = \bar{v}^B$, as well as $\bar{u} = \bar{u}^C$ or $\bar{u} = \bar{u}^B$.

Proposition 4.4. *Define an $(a+b+1) \times (a+b+1)$ matrix $\mathcal{N}^{(s)}$ as follows*

$$\begin{aligned} \mathcal{N}_{j,k}^{(s)} &= \mathcal{N}_{j,k}, \quad j = 1, \dots, a+b; \\ \mathcal{N}_{a+b+1,k}^{(s)} &= Y_k^{(s)}. \end{aligned} \quad (4.18)$$

Here the matrix \mathcal{N} is given by (4.4), (4.5). Then

$$\mathcal{F}_{a,b}^{(s,s)}(z) = (-1)^b \mathcal{H}_{a',b} \cdot \det_{a+b+1} \mathcal{N}^{(s)}, \quad (4.19)$$

where $\mathcal{H}_{a',b}$ is given by (4.2).

Remark. One should remember that in the case of the form factors $\mathcal{F}_{a,b}^{(s,s)}(z)$ one has $a' = a$, while $a' = a+1$ in the case of the form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$. Therefore the function $\mathcal{H}_{a',b}$ in (4.19) is given by (4.2), where one should set $a' = a$. The same remark concerns the entries of the matrix $\mathcal{N}^{(s)}$.

We prove this proposition in section 6, reducing the representation (4.19) to the formulas obtained in [33]. However before doing this we would like to mention that similarly to the $GL(2)$ -case representation (4.19) implies several alternative determinant formulas for the form

factors of the diagonal entries of the monodromy matrix. They can be obtained from (4.19) via the morphisms (2.17) and (2.20).

It is also worth mentioning that

$$\begin{aligned} \sum_{s=1}^3 Y_k^{(s)} &= 0, \quad k = 1, \dots, a+b, \\ \sum_{s=1}^3 Y_{a+b+1}^{(s)} &= \frac{\tau(z|\bar{u}, \bar{v})}{f(z, \bar{u})f(\bar{v}, z)}. \end{aligned} \quad (4.20)$$

Therefore the form factor of the transfer matrix reduces to its eigenvalue $\tau(z|\bar{u}, \bar{v})$ multiplied by the minor of the matrix $\mathcal{N}^{(s)}$ built on the first $(a+b)$ rows and columns. This minor vanishes, if $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$ (see [32] and section 6.1), and thus, the form factor of the transfer matrix between different states is equal to zero, as it should be. Otherwise, if $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$, then the form factor of the transfer matrix is equal to the eigenvalue $\tau(z|\bar{u}, \bar{v})$ multiplied by square of the norm of Bethe vector (see section 6.2).

4.3 Form factor of $T_{13}(z)$

The form factors of the matrix element $T_{13}(z)$ is defined as

$$\mathcal{F}_{a,b}^{(1,3)}(z) \equiv \mathcal{F}_{a,b}^{(1,3)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) T_{13}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (4.21)$$

where $a' = a+1$ and $b' = b+1$. As already mentioned, the calculation of this form factor relies on a new method, that will be presented elsewhere. However, to have here a complete overview of the form factors of the $GL(3)$ case, we preview the result. The determinant representation of the $T_{13}(z)$ form factor is similar to the ones for the form factors of the diagonal entries $T_{ss}(z)$. We again combine the sets \bar{u}^B and \bar{v}^C and the parameter z into the set $\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$. However now this set contains $a' + b'$ (that is, $a+b+2$) elements. We also introduce $(a' + b')$ -component vector $Y^{(1,3)}$ as

$$Y_k^{(1,3)} = (-1)^{b'} \frac{r_3(x_k)h(x_k, \bar{v}^B)}{f(x_k, \bar{u}^B)h(\bar{v}^C, x_k)} + \frac{h(\bar{v}^B, x_k)}{h(\bar{v}^C, x_k)}. \quad (4.22)$$

Proposition 4.5. *Define an $(a' + b') \times (a' + b')$ matrix $\mathcal{N}^{(1,3)}$ as follows*

$$\begin{aligned} \mathcal{N}_{j,k}^{(1,3)} &= \mathcal{N}_{j,k}, \quad j = 1, \dots, a' + b; \\ \mathcal{N}_{a'+b',k}^{(1,3)} &= Y_k^{(1,3)}. \end{aligned} \quad (4.23)$$

Here the matrix \mathcal{N} is given by (4.4), (4.5). Then

$$\mathcal{F}_{a,b}^{(1,3)}(z) = (-1)^{b'} \mathcal{H}_{a',b} \cdot \det_{a'+b+1} \mathcal{N}^{(1,3)}, \quad (4.24)$$

where $\mathcal{H}_{a',b}$ is given by (4.2).

Note that one can obtain an alternative determinant representation for the form factor $\mathcal{F}_{a,b}^{(1,3)}(z)$ applying the mapping (2.20) to the result (4.24). In its turn, the application of the antimorphism (2.17) to (4.24) leads us to a determinant representation for the form factor $\mathcal{F}_{a,b}^{(3,1)}(z)$.

5 Calculation of form factors

As we have mentioned already, the determinant representation for the scalar product of on-shell Bethe vector and generic Bethe vector plays a key role in calculating form factors in $GL(2)$ -based models. In the case of the $GL(3)$ group, an analog of such determinant representation is not known. Therefore calculating the form factor becomes much more involved. The reader can find the details of these calculations in papers [32, 33, 34]. Here we give only a general description of the method that we have used in the papers above.

The study of form factors is based on an explicit representation for the scalar products of Bethe vectors obtained in [38, 39, 40]. The scalar product is defined as

$$S_{a,b} \equiv S_{a,b}(\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B). \quad (5.1)$$

Here the Bethe parameters $\{\bar{u}^C, \bar{v}^C\}$ and $\{\bar{u}^B, \bar{v}^B\}$ are supposed to be generic complex numbers. The representation obtained in [38] describes the scalar product as a sum over partitions of Bethe parameters into subsets (so called *sum formula*). Generically this representation is not reducible to a more compact form. However, when calculating the form factors, we deal with very particular scalar products, where most of the parameters satisfy the Bethe equations (2.10). In such cases, one can reduce this sum over partitions to a single determinant.

Consider, for example, the form factor of the operator $T_{12}(z)$. The action of $T_{12}(z)$ onto $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ is (see [35])

$$T_{12}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = f(\bar{v}^B, z) \mathbb{B}^{a+1,b}(\{\bar{u}^B, z\}; \bar{v}^B) \quad (5.2)$$

$$+ \sum_{i=1}^b g(z, v_i^B) f(\bar{v}_i^B, v_i^B) \mathbb{B}^{a+1,b}(\{\bar{u}^B, z\}; \{\bar{v}_i^B, z\}). \quad (5.3)$$

Thus, the form factor of $T_{12}(z)$ is equal to

$$\mathcal{F}_{a,b}^{(1,2)}(z) = f(\bar{v}^B, z) S_{a+1,b}(\bar{u}^C, \bar{v}^C; \{\bar{u}^B, z\}, \bar{v}^B) \quad (5.4)$$

$$+ \sum_{i=1}^b g(z, v_i^B) f(\bar{v}_i^B, v_i^B) S_{a+1,b}(\bar{u}^C, \bar{v}^C; \{\bar{u}^B, z\}, \{\bar{v}_i^B, z\}), \quad (5.5)$$

and we have reduced the original problem to the calculation of the scalar products, where only z is an arbitrary complex number, while other variables satisfy Bethe equations (2.10).

Formally, other form factors can be calculated in a similar manner. It was proved in [35] that the action of the monodromy matrix entries on Bethe vectors reduces to a linear combination of the last ones. Thus, the form factors of $T_{ij}(z)$ always can be expressed in terms of linear combination of scalar products. However, every specific case has its own peculiarities. In particular, as we have explained above, there is no need to perform a special consideration of form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i - j| = 1$, as all of them can be obtained from $\mathcal{F}_{a,b}^{(1,2)}(z)$ via the mappings (2.17), (2.20).

The form factors of the diagonal operators $T_{ss}(z)$, also can be calculated in the framework of the scheme described above. However the action of $T_{ss}(z)$ onto $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ is much more involved than (5.3). In particular, it contains a double sum over the Bethe parameters. This

fact makes the straightforward calculation of $\mathcal{F}_{a,b}^{(s,s)}(z)$ very complex from a technical viewpoint. Therefore, in the case of form factors of the diagonal entries of the monodromy matrix, it is more convenient to apply a special trick, based on the use of the *twisted transfer matrix*. We describe this method in the next subsection.

Finally, the calculation of form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i - j| = 2$ also should be included into the general scheme. However, in this case we did not succeed to perform the summation over partitions to a single determinant, because of technical problems. It seems rather strange, because the action of the operator $T_{13}(z)$ on the Bethe vectors is the most simple

$$T_{13}(z)\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \mathbb{B}^{a+1,b+1}(\{\bar{u}^B, z\}; \{\bar{v}^B, z\}), \quad (5.6)$$

and therefore the form factor of $T_{13}(z)$ is given by a single scalar product:

$$\mathcal{F}_{a,b}^{(1,3)}(z) = S_{a+1,b+1}(\bar{u}^C, \bar{v}^C; \{\bar{u}^B, z\}, \{\bar{v}^B, z\}). \quad (5.7)$$

Nevertheless, in spite of this simplicity the method of calculation of the sums over partitions of Bethe parameters arising in (5.7) is not developed for today. Therefore for the study the form factor $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i - j| = 2$ we use another approach, which will be described in a separate publication. Here we would like to mention only that the form factors $\mathcal{F}_{a,b}^{(1,3)}(z)$ and $\mathcal{F}_{a,b}^{(3,1)}(z)$ are related by the mapping (2.17).

5.1 Twisted transfer matrix

$GL(3)$ -invariance of R -matrix (2.1) means that $[\hat{\kappa}_1 \hat{\kappa}_2, R_{12}] = 0$ for arbitrary $\hat{\kappa} \in GL(3)$. It is easy to see [41, 42, 38, 43] that due to this property a *twisted monodromy matrix* $\hat{\kappa}T(w)$ satisfies the algebra (2.2). If the matrix $\hat{\kappa}T(w)$ possesses the same pseudovacuum and dual pseudovacuum vectors as the original matrix $T(w)$, then one can apply all the tools of the nested algebraic Bethe ansatz to the twisted monodromy matrix. In particular, one can find the spectrum of the twisted transfer matrix $\text{tr } \hat{\kappa}T(w)$. Its eigenvectors are called twisted on-shell Bethe vectors (or simply twisted on-shell vectors).

Consider a matrix $\hat{\kappa} = \text{diag}(\kappa_1, \kappa_2, \kappa_3)$, where κ_i are arbitrary complex numbers. Obviously, the corresponding twisted monodromy matrix has the same pseudovacuum and dual pseudovacuum vectors. Actually, the multiplication of $T(w)$ by $\hat{\kappa}$ reduces to the replacement of the original eigenvalues $\lambda_i(w)$ (2.8) by $\kappa_i \lambda_i(w)$. Therefore, like the standard on-shell vectors, the twisted on-shell vectors can be parameterized by a set of complex parameters satisfying the twisted Bethe equations. The last ones have the form (2.10), where one should replace $r_k(z)$ by $r_k(z) \kappa_k / \kappa_2$. Below we will need these equations in the logarithmic form. Namely, let

$$\Phi_j = \log r_1(u_j) - \log \left(\frac{f(u_j, \bar{u}_j)}{f(\bar{u}_j, u_j)} \right) - \log f(\bar{v}, u_j), \quad j = 1, \dots, a, \quad (5.8)$$

$$\Phi_{a+j} = \log r_3(v_j) - \log \left(\frac{f(\bar{v}_j, v_j)}{f(v_j, \bar{v}_j)} \right) - \log f(v_j, \bar{u}), \quad j = 1, \dots, b. \quad (5.9)$$

Then the system of twisted Bethe equations has the form

$$\begin{aligned} \Phi_j &= \log \kappa_2 - \log \kappa_1 + 2\pi i \ell_j, & j &= 1, \dots, a, \\ \Phi_{a+j} &= \log \kappa_2 - \log \kappa_3 + 2\pi i m_j, & j &= 1, \dots, b, \end{aligned} \quad (5.10)$$

where ℓ_j and m_j are some integers. The Jacobian of (5.8) and (5.9) is closely related to the norm of the on-shell Bethe vector and the average values of the operators $T_{ss}(z)$ [33].

Using the notion of the twisted transfer matrix one can calculate the form factors of the diagonal entries of the monodromy matrix. Consider the expectation value

$$Q_{\bar{\kappa}}(z) = \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) (\text{tr } \hat{\kappa} T(z) - \text{tr } T(z)) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (5.11)$$

where $\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B)$ are twisted and standard on-shell vectors respectively. Here and below we denote $\bar{\kappa} = \{\kappa_1, \kappa_2, \kappa_3\}$. Obviously

$$Q_{\bar{\kappa}}(z) = \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \sum_{j=1}^3 (\kappa_j - 1) T_{jj}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (5.12)$$

and therefore

$$\left. \frac{dQ_{\bar{\kappa}}(z)}{d\kappa_s} \right|_{\bar{\kappa}=1} = \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \Big|_{\bar{\kappa}=1} T_{ss}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B). \quad (5.13)$$

Here $\bar{\kappa} = 1$ means that $\kappa_i = 1$ for $i = 1, 2, 3$. Observe that after setting $\bar{\kappa} = 1$ the vector $\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C)$ turns into the standard on-shell vector $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C)$. Hence, we obtain the form factor of $T_{ss}(z)$ in the r.h.s. of (5.13)

$$\left. \frac{dQ_{\bar{\kappa}}(z)}{d\kappa_s} \right|_{\bar{\kappa}=1} = \mathcal{F}_{a,b}^{(s,s)}(z | \bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (5.14)$$

On the other hand

$$Q_{\bar{\kappa}}(z) = (\tau_{\bar{\kappa}}(z | \bar{u}^C; \bar{v}^C) - \tau(z | \bar{u}^B; \bar{v}^B)) \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \quad (5.15)$$

where $\tau(z | \bar{u}^B; \bar{v}^B)$ is the eigenvalue of $\text{tr } T(z)$ (2.12), while $\tau_{\bar{\kappa}}(z | \bar{u}^C; \bar{v}^C)$ is the eigenvalues of the twisted transfer matrix $\text{tr } \hat{\kappa} T(z)$:

$$\tau_{\bar{\kappa}}(z) \equiv \tau_{\bar{\kappa}}(z | \bar{u}, \bar{v}) = \kappa_1 r_1(z) f(\bar{u}, z) + \kappa_2 f(z, \bar{u}) f(\bar{v}, z) + \kappa_3 r_3(z) f(z, \bar{v}). \quad (5.16)$$

Thus, we obtain

$$\mathcal{F}_{a,b}^{(s,s)}(z) = \left. \frac{d}{d\kappa_s} \left[(\tau_{\bar{\kappa}}(z | \bar{u}^C; \bar{v}^C) - \tau(z | \bar{u}^B; \bar{v}^B)) \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \right] \right|_{\bar{\kappa}=1}, \quad (5.17)$$

and we see that the form factors $\mathcal{F}_{a,b}^{(s,s)}(z)$ can be calculated as κ -derivatives of the scalar product between twisted on-shell and standard on-shell vectors.

6 Proof of proposition 4.4

In this section we prove proposition 4.4. More precisely, we show that the determinant representations given by proposition 4.4 are equivalent to the ones obtained in [33].

Dealing with the form factors of diagonal entries $T_{ss}(z)$ one should distinguish between two cases:

- $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$;
- $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$.

We consider these two cases separately.

6.1 Proof for different states

In this section $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$. It means that there exists at least one $w \in \{\bar{u}^C, \bar{v}^C\}$, such that $w \notin \{\bar{u}^B, \bar{v}^B\}$. Then

$$\mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \Big|_{\bar{\kappa}=1} = 0, \quad (6.1)$$

as a product of two eigenstates corresponding to the different eigenvalues of the transfer matrix. Hence, the κ -derivative in (5.17) should be applied only to this scalar product. We obtain

$$\mathcal{F}_{a,b}^{(s,s)}(z) = (\tau(z|\bar{u}^C; \bar{v}^C) - \tau(z|\bar{u}^B; \bar{v}^B)) \frac{d}{d\kappa_s} \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \Big|_{\bar{\kappa}=1}. \quad (6.2)$$

The κ -derivatives of the scalar product between twisted on-shell and standard on-shell vectors were calculated in [33]. Let us describe this result.

First of all we introduce an $(a+b)$ -component vector Ω as

$$\begin{aligned} \Omega_j &= \frac{g(u_j^C, \bar{u}_j^C)}{g(u_j^C, \bar{u}_j^B)}, & j = 1, \dots, a, \\ \Omega_{a+j} &= \frac{g(v_j^B, \bar{v}_j^B)}{g(v_j^B, \bar{v}_j^C)}, & j = 1, \dots, b. \end{aligned} \quad (6.3)$$

It is easy to see that since $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$, this vector has at least one non-zero component. Without loss of generality we assume that $\Omega_{a+b} \neq 0$. Then the result for the κ -derivative of the scalar product reads

$$\frac{d}{d\kappa_s} \mathbb{C}_{\bar{\kappa}}^{a,b}(\bar{u}^C; \bar{v}^C) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) \Big|_{\bar{\kappa}=1} = \Omega_{a+b}^{-1} H_{a,b} \widehat{\mathcal{N}}_{a+b,a+b+1}^{(s)}. \quad (6.4)$$

Here

$$H_{a,b} = \frac{(-1)^b \mathcal{H}_{a,b}}{f(z, \bar{u}^B) f(\bar{v}^C, z)} = \frac{h(\bar{w}, \bar{u}^B) h(\bar{v}^C, \bar{w})}{h(\bar{v}^C, \bar{u}^B)} \Delta'_a(\bar{u}^C) \Delta'_b(\bar{v}^B) \Delta_{a+b}(\bar{w}), \quad (6.5)$$

where $\mathcal{H}_{a,b}$ is given by (4.2) and $\bar{w} = \{\bar{u}^B, \bar{v}^C\}$. The factor $\widehat{\mathcal{N}}_{a+b,a+b+1}^{(s)}$ in (6.4) is the cofactor to the element $\mathcal{N}_{a+b,a+b+1}^{(s)}$ of the matrix $\mathcal{N}^{(s)}$ (4.18)

$$\widehat{\mathcal{N}}_{a+b,a+b+1}^{(s)} = - \det_{\substack{j \neq a+b \\ k \neq a+b+1}} \mathcal{N}_{j,k}^{(s)}. \quad (6.6)$$

Let us reproduce this result starting from the determinant representation (4.19). First of all we give the entries of the matrix \mathcal{N} more explicitly

$$\mathcal{N}_{j,k} = (-1)^{a'-1} t(u_j^C, x_k) \frac{r_1(x_k) h(\bar{u}^C, x_k)}{f(\bar{v}^C, x_k) h(x_k, \bar{u}^B)} + t(x_k, u_j^C) \frac{h(x_k, \bar{u}^C)}{h(x_k, \bar{u}^B)}, \quad j = 1, \dots, a', \quad (6.7)$$

$$\mathcal{N}_{a'+j,k} = (-1)^{b-1} t(x_k, v_j^B) \frac{r_3(x_k) h(x_k, \bar{v}^B)}{f(x_k, \bar{u}^B) h(\bar{v}^C, x_k)} + t(v_j^B, x_k) \frac{h(\bar{v}^B, x_k)}{h(\bar{v}^C, x_k)}, \quad j = 1, \dots, b. \quad (6.8)$$

Note that in the case under consideration $a' = a$ and $b' = b$. We use, however, the symbol a' in (6.7), (6.8), because in this form the equations above are still valid for form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$, where $a' = a + 1$.

Let

$$S(x_k) = \sum_{j=1}^{a'} \Omega_j \mathcal{N}_{j,k} + \sum_{j=1}^b \Omega_{a'+j} \mathcal{N}_{a'+j,k}. \quad (6.9)$$

Then using (A.1) one can easily find

$$S(x_k) = \frac{\tau(x_k | \bar{u}^C, \bar{v}^C) - \tau(x_k | \bar{u}^B, \bar{v}^B)}{f(\bar{v}^C, x_k) f(x_k, \bar{u}^B)}. \quad (6.10)$$

It is straightforward to check that $S(u_k^B) = S(v_k^C) = 0$ due to the Bethe equations. In fact, one can see this without any calculations. Indeed, the Bethe equations are equivalent to the statement that the function $\tau(x_k | \bar{u}, \bar{v})$ has no poles in the points $x_k = u_j$ and $x_k = v_j$ (see Remark on the page 5). Then the factor $f^{-1}(\bar{v}^C, x_k) f^{-1}(x_k, \bar{u}^B)$ immediately yields the equalities $S(u_k^B) = S(v_k^C) = 0$.

Now we multiply the first $(a + b - 1)$ rows of the matrix $\mathcal{N}^{(s)}$ by the factors Ω_j / Ω_{a+b} and add them to the $(a + b)$ -th row. Then we obtain a modified $(a + b)$ -th row with the components

$$\begin{aligned} \mathcal{N}_{a+b,k}^{(s),\text{mod}} &= 0, \quad k = 1, \dots, a + b, \\ \mathcal{N}_{a+b,a+b+1}^{(s),\text{mod}} &= \Omega_{a+b}^{-1} \frac{\tau(z | \bar{u}^C, \bar{v}^C) - \tau(z | \bar{u}^B, \bar{v}^B)}{f(\bar{v}^C, z) f(z, \bar{u}^B)}. \end{aligned} \quad (6.11)$$

The determinant $\det \mathcal{N}^{(s)}$ reduces to the product of the element $\mathcal{N}_{a+b,a+b+1}^{(s),\text{mod}}$ by the corresponding cofactor, and we arrive at

$$\det_{a+b+1} \mathcal{N}^{(s)} = \Omega_{a+b}^{-1} \frac{\tau(z | \bar{u}^C, \bar{v}^C) - \tau(z | \bar{u}^B, \bar{v}^B)}{f(\bar{v}^C, z) f(z, \bar{u}^B)} \hat{\mathcal{N}}_{a+b,a+b+1}^{(s)}. \quad (6.12)$$

We would like to draw the reader's attention that the matrix element $Y_{a+b+1}^{(s)}$ has disappeared from the game. Substituting this result into (4.19) we immediately reproduce (6.4).

6.2 Proof for the same states

In this section $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$ and we set $\bar{u}^C = \bar{u}^B = \bar{u}$ and $\bar{v}^C = \bar{v}^B = \bar{v}$. In this case

$$\tau_{\bar{\kappa}}(z | \bar{u}^C; \bar{v}^C) - \tau(z | \bar{u}^B; \bar{v}^B) = 0, \quad \text{at} \quad \bar{\kappa} = 1; \quad \bar{u}^C = \bar{u}^B = \bar{u}; \quad \bar{v}^C = \bar{v}^B = \bar{v}, \quad (6.13)$$

hence, the κ -derivative in (5.17) should act only on the difference of the eigenvalues $\tau_{\bar{\kappa}}$ and τ . Then we find

$$\mathcal{F}^{(s,s)}(z | \bar{u}, \bar{v}; \bar{u}, \bar{v}) = \|\mathbb{B}^{a,b}(\bar{u}; \bar{v})\|^2 \frac{d\tau_{\bar{\kappa}}(z | \bar{u}^C; \bar{v}^C)}{d\kappa_s} \Big|_{\bar{\kappa}=1}, \quad (6.14)$$

and one should set $\bar{u}^C = \bar{u}$ and $\bar{v}^C = \bar{v}$ after taking the derivative of $\tau_{\bar{\kappa}}(z|\bar{u}^C; \bar{v}^C)$ with respect to κ_s . Below in this section we always assume that the condition $\bar{\kappa} = 1$ automatically yields $\bar{u}^C = \bar{u}^B = \bar{u}$ and $\bar{v}^C = \bar{v}^B = \bar{v}$.

The square of the norm of on-shell Bethe vector $\|\mathbb{B}^{a,b}(\bar{u}; \bar{v})\|^2$ was calculated in [38, 32]. It is proportional to the minor of the matrix $\mathcal{N}^{(s)}$ built on the first $(a+b)$ rows and columns⁴:

$$\|\mathbb{B}^{a,b}(\bar{u}; \bar{v})\|^2 = H_{a,b} \det_{a+b} \mathcal{N}, \quad (6.15)$$

where $H_{a,b}$ is given by (6.5) at $\bar{u}^C = \bar{u}^B = \bar{u}$ and $\bar{v}^C = \bar{v}^B = \bar{v}$.

Let us give explicitly the entries of the matrix \mathcal{N} in the case $\bar{u}^C = \bar{u}^B = \bar{u}$ and $\bar{v}^C = \bar{v}^B = \bar{v}$ (see [38, 32]). For $j, k = 1, \dots, a$ we have

$$\mathcal{N}_{j,k} = \delta_{jk} \left(-c \log' r_1(u_k) - \sum_{\ell=1}^a \frac{2c^2}{u_{k\ell}^2 - c^2} + \sum_{m=1}^b t(v_m, u_k) \right) + \frac{2c^2}{u_{jk}^2 - c^2}, \quad (6.16)$$

where $u_{k\ell} = u_k - u_\ell$. The entries of the second diagonal block are

$$\mathcal{N}_{a+j, a+k} = \delta_{jk} \left(c \log' r_3(v_k) - \sum_{m=1}^b \frac{2c^2}{v_{km}^2 - c^2} + \sum_{\ell=1}^a t(v_k, u_\ell) \right) + \frac{2c^2}{v_{jk}^2 - c^2}, \quad (6.17)$$

where $v_{km} = v_k - v_m$ and $j, k = 1, \dots, b$. The antidiagonal blocks have more simple structure

$$\mathcal{N}_{j, a+k} = t(v_k, u_j), \quad j = 1, \dots, a, \quad k = 1, \dots, b, \quad (6.18)$$

$$\mathcal{N}_{a+j, k} = t(v_j, u_k), \quad j = 1, \dots, b, \quad k = 1, \dots, a. \quad (6.19)$$

Observe that the matrix \mathcal{N} is symmetric: $\mathcal{N}_{jk} = \mathcal{N}_{kj}$. It is also easy to check (see [32]) that

$$\begin{aligned} \mathcal{N}_{j,k} &= -c \frac{\partial \Phi_j}{\partial u_k}, & j = 1, \dots, a+b, \quad k = 1, \dots, a; \\ \mathcal{N}_{j, a+k} &= c \frac{\partial \Phi_j}{\partial v_k}, & j = 1, \dots, a+b, \quad k = 1, \dots, b, \end{aligned} \quad (6.20)$$

where Φ_j is given by (5.8), (5.9).

Let us reproduce the result (6.14) starting from the representation (4.19). The entries of the matrix $\mathcal{N}_{j,k}^{(s)}$ with $j, k = 1, \dots, a+b$ coincide with the ones defined in (6.16)–(6.19). In the last row we have

$$\begin{aligned} \mathcal{N}_{a+b+1, k}^{(s)} &= Y_k^{(s)} = \delta_{s2} - \delta_{s1}, & k = 1, \dots, a; \\ \mathcal{N}_{a+b+1, k}^{(s)} &= Y_k^{(s)} = \delta_{s2} - \delta_{s3}, & k = a+1, \dots, b. \end{aligned} \quad (6.21)$$

⁴Pay attention that this minor does not depend on s .

Finally, the last column has the components

$$\begin{aligned}
\mathcal{N}_{j,a+b+1}^{(s)} &= \frac{c}{f(z, \bar{u})f(\bar{v}, z)} \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial u_j}, & j = 1, \dots, a; \\
\mathcal{N}_{a+j,a+b+1}^{(s)} &= -\frac{c}{f(z, \bar{u})f(\bar{v}, z)} \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial v_j}, & j = 1, \dots, b \\
\mathcal{N}_{a+b+1,a+b+1}^{(s)} &= \frac{1}{f(z, \bar{u})f(\bar{v}, z)} \frac{\partial \tau_{\bar{\kappa}}(z|\bar{u}^C, \bar{v}^C)}{\partial \kappa_s} \Big|_{\bar{\kappa}=1}.
\end{aligned} \tag{6.22}$$

Thus, we have described the $(a+b+1) \times (a+b+1)$ matrix $\mathcal{N}^{(s)}$ in the limit $\bar{u}^C = \bar{u}^B = \bar{u}$ and $\bar{v}^C = \bar{v}^B = \bar{v}$. Let us show that $\det \mathcal{N}^{(s)}$ can be reduced to the determinant of the $(a+b) \times (a+b)$ block of this matrix given by (6.16)–(6.19). For this we introduce three $(a+b)$ -component vectors $\tilde{\Omega}_j^{(s)}$ as

$$\begin{aligned}
\tilde{\Omega}_j^{(s)} &= \frac{1}{c} \frac{du_j^C}{d\kappa_s} \Big|_{\bar{\kappa}=1}, & j = 1, \dots, a, \\
\tilde{\Omega}_{a+j}^{(s)} &= -\frac{1}{c} \frac{dv_j^C}{d\kappa_s} \Big|_{\bar{\kappa}=1}, & j = 1, \dots, b.
\end{aligned} \tag{6.23}$$

It is easy to show that

$$\sum_{j=1}^{a+b+1} \tilde{\Omega}_j^{(s)} \mathcal{N}_{j,k}^{(s)} = 0, \quad k = 1, \dots, a+b. \tag{6.24}$$

Indeed, differentiating the system of twisted Bethe equations (5.10) with respect to κ_s at $\bar{\kappa} = 1$ we obtain

$$\sum_{\ell=1}^a \frac{\partial \Phi_j}{\partial u_\ell} \frac{du_\ell^C}{d\kappa_s} \Big|_{\bar{\kappa}=1} + \sum_{m=1}^b \frac{\partial \Phi_j}{\partial v_m} \frac{dv_m^C}{d\kappa_s} \Big|_{\bar{\kappa}=1} = Y_k^{(s)}. \tag{6.25}$$

Taking into account (6.20) and the symmetry of the matrix $\mathcal{N}_{j,k}^{(s)}$ for $j, k = 1, \dots, a+b$ we immediately arrive at (6.24). Thus, adding to the last row of the matrix $\mathcal{N}_{j,k}^{(s)}$ all other rows multiplied by the coefficients $\tilde{\Omega}_j^{(s)}$ we obtain zeros everywhere except the element $j, k = a+b+1$, where we have

$$\begin{aligned}
\sum_{j=1}^{a+b+1} \tilde{\Omega}_j^{(s)} \mathcal{N}_{j,a+b+1}^{(s)} &= \frac{1}{f(z, \bar{u})f(\bar{v}, z)} \left\{ \frac{\partial \tau(z|\bar{u}^C, \bar{v}^C)}{\partial \kappa_s} \Big|_{\bar{\kappa}=1} \right. \\
&\quad \left. + \sum_{\ell=1}^a \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial u_\ell} \frac{du_\ell^C}{d\kappa_s} \Big|_{\bar{\kappa}=1} + \sum_{m=1}^b \frac{\partial \tau(z|\bar{u}, \bar{v})}{\partial v_m} \frac{dv_m^C}{d\kappa_s} \Big|_{\bar{\kappa}=1} \right\} = \frac{d\tau(z|\bar{u}^C, \bar{v}^C)}{d\kappa_s} \Big|_{\bar{\kappa}=1}.
\end{aligned} \tag{6.26}$$

Thus, we obtain

$$\mathcal{F}_{a,b}^{(s,s)}(z|\bar{u}, \bar{v}; \bar{u}, \bar{v}) = \frac{d\tau(z|\bar{u}, \bar{v})}{d\kappa_s} \Big|_{\bar{\kappa}=1} \cdot H_{a,b} \det_{a+b} \mathcal{N}. \tag{6.27}$$

Comparing (6.27) with (6.15) we arrive at the representation (6.14).

7 Discussions

In this paper we considered the form factors of the monodromy matrix entries in the models with $GL(3)$ -invariant R -matrix and obtained determinant representations for them. The question arises of generalizing the results obtained to the models with the symmetry group of a higher rank. For this it is useful to compare the structure of the determinant formulas for the models with $GL(2)$ and $GL(3)$ symmetry.

For $GL(3)$ -based models all the representations have similar structure and are based on the determinants of the matrix \mathcal{N} or $\tilde{\mathcal{N}}$ (the last one can be obtained from \mathcal{N} by the replacement $\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}$). In these matrices all rows and columns are associated with one of the Bethe parameters or with the external variable z . Say, in the matrix \mathcal{N} the first a columns correspond to the set \bar{u}^B , the next b columns correspond to the set \bar{v}^C , and the last column is associated with the variable z . The rows of this matrix are associated with the parameters \bar{u}^C , \bar{v}^B . For the form factor of the diagonal entries, as well as for the operator $T_{13}(z)$, the matrix \mathcal{N} has an additional row.

It is hardly possible to predict such the structure based on the results obtained for the models possessing $GL(2)$ symmetry. One could expect that, for example, the columns of the matrices should correspond to the parameters of one Bethe vector (say, $\{\bar{u}^B, \bar{v}^B\}$), while the rows should correspond to the parameters of another Bethe vector (in this case, $\{\bar{u}^C, \bar{v}^C\}$). We see, however, that it is not the case, and one should ‘mix’ the parameters from different Bethe vectors in order to label the rows and the columns.

Such mixing of the Bethe parameters makes very problematic a straightforward generalization of our results to the models with $GL(N)$ -symmetry with $N > 3$. There exists also one more argument to rule out a simple generalization of these results to the symmetry groups of higher rank. We see that the matrix whose determinant describe form factors, have a block structure

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_\ell & & \\ - & - & - \\ \mathcal{N}_r & & \end{pmatrix}, \quad \text{where} \quad \begin{aligned} (\mathcal{N}_\ell)_{j,k} &\sim \frac{\partial \tau(x_k | \bar{u}^C, \bar{v}^C)}{\partial u_j^C}, \\ (\mathcal{N}_r)_{j,k} &\sim \frac{\partial \tau(x_k | \bar{u}^B, \bar{v}^B)}{\partial v_j^B}. \end{aligned} \quad (7.1)$$

The upper and lower blocks are proportional to the Jacobians of the transfer matrix eigenvalues on the left and the right Bethe vectors respectively. On the other hand the block structure is also related to the fact that Bethe vectors depend on two sets of parameters. However, in the case of the $GL(N)$ group, Bethe vectors depend on $N - 1$ sets of variables [29]. Hence, it is natural to expect that if there are determinant representations for form factors in the models with symmetry group, for example $GL(4)$, then the corresponding matrices should have a block structure 3×3 . At the same time we still have only two vectors and, hence, only two eigenvalues.

Of course, the arguments above do not mean that determinant representations for form factors do not exist in the models with $GL(N)$ -invariant R -matrix. These arguments can only tell that the determinant representations based on the Jacobians of the transfer matrix eigenvalues are hardly possible for models with higher symmetry group. However, on the other hand, we can not exclude the existence of determinant representations having different structure.

Concluding this paper we would like to say few words about possible applications. One of them immediately arises for the quantum models admitting explicit solution of the quantum inverse scattering problem [16, 21]. In particular, one has the following representation for the

local operators in the $SU(3)$ -invariant XXX Heisenberg chain:

$$E_m^{\alpha,\beta} = (\text{tr } T(0))^{m-1} T_{\beta\alpha}(0) (\text{tr } T(0))^{-m}. \quad (7.2)$$

Here $E_m^{\alpha,\beta}$, $\alpha, \beta = 1, 2, 3$, is an elementary unit $((E^{\alpha,\beta})_{jk} = \delta_{j\alpha}\delta_{k\beta})$ associated with the m -th site of the chain. Since the action of the transfer matrix $\text{tr } T(0)$ on on-shell Bethe vectors is trivial, we see that the form factors of $E_m^{\alpha,\beta}$ are proportional to those of $T_{\beta\alpha}$

$$\mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) E_m^{\alpha,\beta} \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B) = \frac{\tau^{m-1}(0|\bar{u}^C, \bar{v}^C)}{\tau^m(0|\bar{u}^B, \bar{v}^B)} \mathcal{F}_{a,b}^{(\beta,\alpha)}(0|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B). \quad (7.3)$$

Thus, if we have an explicit and compact representations for form factors of $T_{\beta,\alpha}$, we can study the problem of two-point and multi-point correlation functions, expanding them into series with respect to the form factors.

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A Summation formulas

In this section we prove several identities for the vector Ω introduced in (6.3).

Proposition A.1. *Let Ω is defined as in (6.3). Then*

$$\begin{aligned} \sum_{j=1}^a t(u_j^C, z) \Omega_j &= \frac{h(\bar{u}^B, z)}{h(\bar{u}^C, z)} \left(1 - \frac{f(\bar{u}^C, z)}{f(\bar{u}^B, z)} \right), \\ \sum_{j=1}^a t(z, u_j^C) \Omega_j &= \frac{h(z, \bar{u}^B)}{h(z, \bar{u}^C)} \left(\frac{f(z, \bar{u}^C)}{f(z, \bar{u}^B)} - 1 \right), \\ \sum_{j=1}^b t(v_j^B, z) \Omega_{j+a} &= \frac{h(\bar{v}^C, z)}{h(\bar{v}^B, z)} \left(1 - \frac{f(\bar{v}^B, z)}{f(\bar{v}^C, z)} \right), \\ \sum_{j=1}^b t(z, v_j^B) \Omega_{j+a} &= \frac{h(z, \bar{v}^C)}{h(z, \bar{v}^B)} \left(\frac{f(z, \bar{v}^B)}{f(z, \bar{v}^C)} - 1 \right). \end{aligned} \quad (\text{A.1})$$

All the identities above can be proved in a similar way. Consider, for example, the first identity.

Proof. Let

$$\sum_{j=1}^a t(u_j^C, z) \Omega_j = W(z). \quad (\text{A.2})$$

The sum in the l.h.s. of (A.2) can be computed by means of an auxiliary integral

$$I = \frac{1}{2\pi i} \oint_{|\omega|=R \rightarrow \infty} \frac{c d\omega}{(\omega - z)(\omega - z + c)} \prod_{\ell=1}^a \frac{\omega - u_{\ell}^B}{\omega - u_{\ell}^C}. \quad (\text{A.3})$$

The integral is taken over the anticlockwise oriented contour $|\omega| = R$ and we consider the limit $R \rightarrow \infty$. Then $I = 0$, because the integrand behaves as $1/\omega^2$ at $\omega \rightarrow \infty$. On the other hand the same integral is equal to the sum of the residues within the integration contour. Obviously the sum of the residues at $\omega = u_{\ell}^C$ gives $W(z)$. There are also two additional poles at $\omega = z$ and $\omega = z - c$. Then we have

$$I = 0 = W(z) - \prod_{\ell=1}^a \frac{z - u_{\ell}^B - c}{z - u_{\ell}^C - c} + \prod_{\ell=1}^a \frac{z - u_{\ell}^B}{z - u_{\ell}^C}. \quad (\text{A.4})$$

From this we obtain the first identity (A.1)

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